Quantum Mechanics and the Interpretation of the Orthomodular Square of Opposition

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Abstract

In this paper we analyze and discuss the historical and philosophical development of the notion of logical possibility focusing on its specific meaning in classical and quantum mechanics. Taking into account the logical structure of quantum theory we continue our discussion regarding the Aristotelian Square of Opposition in orthomodular structures enriched with a monadic quantifier [7]. Finally, we provide an interpretation of the *Orthomodular Square of Opposition* exposing the fact that classical possibility and quantum possibility behave formally in radically different manners.

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1 The Modes of Physical Properties: Actuality and Potentiality

The debate in Pre-Soctratic philosophy is traditionally understood as the contraposition of the Heraclitean and the Eleatic schools of thought [18].

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Heraclitus was considered as defending the theory of flux, a doctrine of permanent motion, change and unstability in the world. This doctrine precluded, as both Plato and Aristotle stressed repeatedly, the impossibility to develop certain knowledge about the world. "This is so because Being, over a lapse of time, has no stability. Everything that it is at this moment changes at the same time, therefore it is not. This coming together of Being and non-Being at one instant is known as the principle of coincidence of opposites." [20, p. 2] In contraposition to the Heraclitean school we find Parmenides as the main character of the Eleatic school. Parmenides, as interpreted also by Plato and Aristotle, taught the non-existence of motion and change in reality, reality being absolutely One, and being absolutely Being. In his famous poem Parmenides stated maybe the earliest intuitive exposition of the *principle of non-contradiction*; i.e. that which *is* can only *be*, that which *is not, cannot be.* In order to dissolve the problem of movement, Aristotle developed a metaphysical scheme in which, through the notions of *actuality* and *potentiality*, he was able to articulate both the Heraclitean and the Eleatic school [1]. On the one hand, potentiality contained the undetermined, contradictory and non-individual realm of existence, on the other, the mode of being of actuality was determined through the logical principles of *existence* and *non-contradiction*; it was through these same principles together with the principle of *identity* that the concept of entity was put forward. Through these principles the notion of entity is capable of unifying, of totalizing in terms of a "sameness", creating certain stability for knowledge to be possible. This representation or transcendent description of the world is considered by many the origin of metaphysical thought. Actuality is then linked directly to metaphysical representation and understood as characterizing a mode of existence independent of observation. This is the way through which metaphysical thought was able to go beyond the *hic et nunc*, creating a world beyond the world, a world of concepts.

Although Aristotle presents at first both actual and potential realms as ontologically equivalent, from chapter 6 of book Θ, he seems to place actuality in the central axis of his architectonic, relegating potentiality to a mere supplementary role. "We have distinguished the various senses of 'prior', and it is clear that actuality is prior to potentiality. [...] For the action is the end, and the actuality is the action. Therefore even the word 'actuality' is derived from 'action', and points to the fulfillment." [1050a17-1050a23] Aristotle then continues to provide arguments in this line which show "[t]hat the good actuality is better and more valuable than the good potentiality." [1051a4-1051a17] But, quite independently of the Aristotelian metaphysical scheme, it could be argued that the idea of potentiality could be developed in order to provide a mode of existence independent of actuality. As we shall see in the following, after modern science discarded almost completely the potential realm, it was quantum theory through some of its authors that potentiality became again a key concept for physics. Wolfgang Pauli had seen this path in relation to the development of quantum mechanics itself. As noted in a letter to C. G. Jung dated 27 February 1953:

> "Science today has now, I believe, arrived at a stage where it can proceed (albeit in a way as yet not at all clear) along the path laid down by Aristotle. The complementarity characteristics of the electron (and the atom) (wave and particle) are in fact 'potential being,' but one of them is always 'actual nonbeing.' That is why one can say that science, being no longer classical, is for the first time a genuine theory of becoming and no longer Platonic." [15, p. 93]

But before arriving to QM let us first analyze the relation between classical physics and the hilemorphic tradition.

2 The Actual Realm and Classical Physics

The importance of potentiality, which was first placed by Aristotle in equal footing to actuality as a mode of existence, was soon diminished in the history of western thought. As we have seen above, it could be argued that the seed of this move was already present in the Aristotelian architectonic, whose focus was clearly placed in the actual realm. The realm of potentiality, as a different (ontological) mode of the being was neglected becoming not more than mere (logical) *possibility*, a process of fulfillment. In relation to the development of physics, the focus and preeminence was also given to actuality. The XVII century division between 'res cogitans' and 'res extensa' played in this respect an important role separating very clearly the realms of actuality and potentiality. The philosophy which was developed after Descartes kept 'res cogitans' (thought) and 'res extensa' (entities as acquired by the senses) as separated realms.[1]

> "Descartes knew the undisputable necessity of the connection, but philosophy and natural science in the following period developed on the

[1] While 'res cogitans', the soul, was related to the *indefinite* realm of potentiality, 'res extensa', i.e. the entities as characterized by the principles of logic, related to the actual.

> basis of the polarity between the 'res cogitans' and the 'res extensa', and natural science concentrated its interest on the 'res extensa'. The influence of the Cartesian division on human thought in the following centuries can hardly be overestimated, but it is just this division which we have to criticize later from the development of physics in our time." [9, p. 73]

This materialistic conception of science based itself on the main idea that extended things exist as being definite, that is, in the actual realm of existence. With modern science the actualist Megarian path was recovered and potentiality dismissed as a problematic and unwanted guest. The transformation from medieval to modern science coincides with the abolition of Aristotelian hilemorphic metaphysical scheme —in terms of potentiality and actuality— as the foundation of knowledge. However, the basic structure of his metaphysical scheme and his logic still remained the basis for correct reasoning. As noted by Verelst and Coecke:

> "Dropping Aristotelian metaphysics, while at the same time continuing to use Aristotelian logic as an empty 'reasoning apparatus' implies therefore loosing the possibility to account for change and motion in whatever description of the world that is based on it. The fact that Aristotelian logic transformed during the twentieth century into different formal, axiomatic logical systems used in today's philosophy and science doesn't really matter, because the fundamental principle, and therefore the fundamental ontology, remained the same ([40], p. xix). This 'emptied' logic actually contains an Eleatic ontology, that allows only for static descriptions of the world." [20, p. 7]

It was Isaac Newton who was able to translate into a closed mathematical formalism both, the ontological presuppositions present in Aristotelian (Eleatic) logic and the materialistic ideal of 'res extensa' together with actuality as its mode of existence. In classical mechanics the representation of the state of the physical system is given by a point in phase space Γ and the physical magnitudes are represented by real functions over Γ. These functions commute in between each others and can be interpreted as possessing definite values independently of measurement, i.e. each function can be interpreted as being actual. The term actual refers here to *preexistence* (within the transcendent representation) and not to the observation *hic et nunc*. Every physical system may be described exclusively by means of its actual properties. The change of the system may be described by the change of its actual properties. Potential or possible properties are considered as the points to which the system might arrive in a future instant of time. As also noted by Dieks:

> "In classical physics the most fundamental description of a physical system (a point in phase space) reflects only the actual, and nothing that is merely possible. It is true that sometimes states involving probabilities occur in classical physics: think of the probability distributions ρ in statistical mechanics. But the occurrence of possibilities in such cases merely reflects our ignorance about what is actual. The statistical states do not correspond to features of the actual system (unlike the case of the quantum mechanical superpositions), but quantify our lack of knowledge of those actual features." [2, p. 124]

Classical mechanics tells us via the equation of motion how the state of the system moves along the curve determined by the initial conditions in Γ and thus, as any mechanical property may be expressed in terms of Γ's variables, how all of them evolve. Moreover, the structure in which actual properties may be organized is the (Boolean) algebra of classical logic.

3 Heisenberg and the Recovery of the Potential Realm

The mechanical description of the world provided by Newton can be sketched in terms of static pictures which provide at each instant of time the set of definite actual properties within a given state of affairs [12, p. 609]. Obviously there is in this description a big debt to the Aristotelian metaphysical scheme. However, the description of motion is then given, not *via* the path from the potential to the actual, from *matter* into *form*, but rather *via* the successions of actual states of affairs; i.e., stable situations, "pictures", constituted by sets of actual properties with definite values. As we discussed above, potentiality becomes then superfluous. With the advenment of modern science and the introduction of mathematical schemes, physics seemed capable of reproducing the evolution of the universe. The idea of an actual state of affairs (i.e. the set of actual properties which characterize a system) supplemented by the dynamics allowed then to imagine a Demon such as that of Laplace capable of knowing the past and future states of the universe. If we could know the actual values at the definite instant of time we could also derive the actual set of properties in the future and the past. As Heisenberg explains, this materialistic conception of science chose actuality as the main aspect of existence:

> "In the philosophy of Aristotle, matter was thought of in the relation between form and matter. All that we perceive in the world of phenomena around us is formed matter. Matter is in itself not a reality but only a possibility, a 'potentia'; it exists only by means of form. In the natural process the 'essence,' as Aristotle calls it, passes over from mere possibility through form into actuality. [...] Then, much later, starting from the philosophy of Descartes, matter was primarily thought of as opposed to mind. There were the two complementary aspects of the world, 'matter' and 'mind,' or, as Descartes put it, the 'res extensa' and the 'res cogitans.' Since the new methodical principles of natural science, especially of mechanics, excluded all tracing of corporeal phenomena back to spiritual forces, matter could be considered as a reality of its own independent of the mind and of any supernatural powers. The 'matter' of this period is 'formed matter,' the process of formation being interpreted as a causal chain of mechanical interactions; it has lost its connection with the vegetative soul of Aristotelian philosophy, and therefore the dualism between matter and form [potential and actual] is no longer relevant. It is this concept of matter which constitutes by far the strongest component in our present use of the word 'matter'." [9, p. 129]

As mentioned above, in classical mechanics the mathematical description of the behavior of a system may be formulated in terms of the set of actual properties. The same treatment can be applied to quantum mechanics. However, the different structure of the physical properties of the system in the new theory determines a change of nature regarding the meaning of possibility and potentiality. Quantum mechanics was related to modality since Born's interpretation of the quantum wave function Ψ in terms of a density of probability. But it was clear from the very beginning that this new quantum possibility was something completely different from that considered in classical theories. "[The] concept of the probability wave [in quantum mechanics] was something entirely new in theoretical physics since Newton. Probability in mathematics or in statistical mechanics means a statement about our degree of knowledge of the actual situation. In throwing dice we do not know the fine details of the motion of our hands which determine the fall of the dice and therefore we say that the probability for throwing a special number is just one in six. The probability wave function, however, meant more than that; it meant a tendency for something." [9, p. 42] It was Heisenberg who went a step further and tried to interpret the wave function in terms of the Aristotelian notion of potentia. Heisenberg argued that the concept of probability wave "was a quantitative version of the old concept of 'potentia' in Aristotelian philosophy. It introduced something standing in the middle between the idea of an event and the actual event, a strange kind of physical reality just in the middle between possibility and reality." According to Heisenberg, the concept of potentiality as a mode of existence has been used implicitly or explicitly in the development of quantum mechanics:

> "I believe that the language actually used by physicists when they speak about atomic events produces in their minds similar notions as the concept of 'potentia'. So physicists have gradually become accustomed to considering the electronic orbits, etc., not as reality but rather as a kind of 'potentia'." [9, p. 156]

In this respect, one of the most interesting examples of an implicit use of these ideas has been provided by Richard Feynmann in his path integral approach [6]. Even though Feynman talks about calculating probabilities, he seems to refer implicitly to of existent potentialities. Why, if not, should we take into account the mutually incompatible paths of the electron in the double-slit experiment? His approach considers every path as existent in the mode of being of potentiality, there where the constrains of actuality cannot be applied. But as we discussed elsewhere [16], Heisenberg's attempt to interpret quantum mechanics with a non-classical conceptual scheme might have been highly compromised by Bohr's own agenda. In any case, we must admit that apart from some few remarks and analogies, Heisenberg's interpretation remained not only incomplete but also unclear in many aspects.

4 Quantum Possibility in the Orthomodular Structure

Elsewhere we have discussed the importance of distinguishing, both from a formal and conceptual level the notion of (classical) possibility —arising in the distributive Boolean structure— from that of quantum possibility

—arising from an orthomodular structure. In order to discuss some interpretational aspects of quantum possibility we first recall from [11, 14] some notions about orthomodular lattices. A *lattice with involution* [10] is an algebra $\langle \mathcal{L}, \vee, \wedge, \neg \rangle$ such that $\langle \mathcal{L}, \vee, \wedge \rangle$ is a lattice and $\neg$ is a unary operation on $\mathcal{L}$ that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. An *orthomodular lattice* is an algebra $\langle \mathcal{L}, \wedge, \vee, \neg, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ that satisfies the following conditions:

1. $\langle \mathcal{L}, \wedge, \vee, \neg, 0, 1 \rangle$ is a bounded lattice with involution,
2. $x \wedge \neg x = 0$.
3. $x \vee (\neg x \wedge (x \vee y)) = x \vee y$

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. Let $\mathcal{L}$ be an orthomodular lattice. *Boolean algebras* are orthomodular lattices satisfying the *distributive law* $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. We denote by $\mathbf{2}$ the Boolean algebra of two elements. Let $\mathcal{L}$ be an orthomodular lattice. An element $c \in \mathcal{L}$ is said to be a *complement* of a iff $a \wedge c = 0$ and $a \vee c = 1$. Given a, b, c in $\mathcal{L}$, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, (a,b,c)D^* hold for all permutations of a, b, c. An element z of $\mathcal{L}$ is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(\mathcal{L})$ the set of all central elements of $\mathcal{L}$ and it is called the *center* of $\mathcal{L}$.

Proposition 4.1 *Let $\mathcal{L}$ be an orthomodular lattice. Then we have:*

1. *$Z(\mathcal{L})$ is a Boolean sublattice of $\mathcal{L}$* [14, Theorem 4.15].
2. *$z \in Z(\mathcal{L})$ iff for each $a \in \mathcal{L}$, $a = (a \wedge z) \vee (a \wedge \neg z)$* [14, Lemma 29.9].

□

In the orthodox formulation of quantum mechanics, a property of (or a proposition about) a quantum system is related to a closed subspace of the Hilbert space $\mathcal{H}$ of its (pure) states or, analogously, to the projector operator onto that subspace. Physical properties of the system are organized in the orthomodular lattice of closed subspaces $\mathcal{L}(\mathcal{H})$ also called *Hilbert lattice*. Let $\mathcal{H}$ be Hilbert space representing a quantum system. Differently from the classical scheme, a physical magnitude $\mathcal{M}$ is represented by an operator $\mathbf{M}$ acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $\mathbf{M} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i\rangle\langle a_i|$ are satisfied. The real numbers a_i are related to the outcomes of measurements of the magnitude $\mathcal{M}$ and projectors $|a_i\rangle\langle a_i|$ to the mentioned properties. Each self-adjoint operator $\mathbf{M}$ has associated a Boolean sublattice $W_{\mathbf{M}}$ of $\mathcal{L}(\mathcal{H})$ which we will refer to as the spectral algebra of the operator $\mathbf{M}$. Assigning values to a physical quantity $\mathcal{M}$ is equivalent to establishing a Boolean homomorphism $v : W_{\mathbf{M}} \rightarrow \mathbf{2}$.

The fact that physical magnitudes are represented by operators on $\mathcal{H}$ that, in general, do not commute has extremely problematic interpretational consequences for it is then difficult to affirm that these quantum magnitudes are *simultaneously preexistent*. In order to restrict the discourse to sets of commuting magnitudes, different Complete Sets of Commuting Operators (CSCO) have to be chosen. This choice has not found until today a clear justification and remains problematic. This feature is called in the literature *quantum contextuality*. The Kochen-Specker theorem (KS theorem for short) rules out the non-contextual assignment of definite values to the physical properties of a quantum system [13]. This may be expressed in terms of valuations over $\mathcal{L}(\mathcal{H})$ in the following manner. We first introduce the concept of global valuation. Let $(W_i)_{i\in I}$ be the family of Boolean sublattices of $\mathcal{L}(\mathcal{H})$. Then a *global valuation* of the physical magnitudes over $\mathcal{L}(\mathcal{H})$ is a family of Boolean homomorphisms $(v_i : W_i \rightarrow \mathbf{2})_{i\in I}$ such that $v_i \mid W_i \cap W_j = v_j \mid W_i \cap W_j$ for each $i, j \in I$. If this global valuation existed, it would allow to give values to all magnitudes at the same time maintaining a *compatibility condition* in the sense that whenever two magnitudes shear one or more projectors, the values assigned to those projectors are the same from every context. The KS theorem, in the algebraic terms, rules out the existence of global valuations when $dim(\mathcal{H}) > 2$ [3, Theorem 3.2]. Contextuality can be directly related to the impossibility to represent a piece of the world as constituted by a set of definite valued properties independently of the choice of the context. This definition makes reference only to the actual realm. But as we know, QM makes probabilistic assertions about measurement results. Therefore, it seems natural to assume that QM does not only deal with actualities but also with possibilities.

Following [4] we delineate a modal extension for orthomodular lattices that allows to formally represent, within the same algebraic structure, actual and possible properties of the system. This allows us to discuss the restrictions posed by the theory itself to the *actualization* of possible properties. Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones (Born rule). In other words, one may predicate truth or falsity of all possibilities at the same time, i.e., possibilities allow an interpretation in a Boolean algebra, i.e., if we refer with $\Diamond P$ to the possibility of P then, $\Diamond P \in Z(\mathcal{L})$. This interpretation of possibility in terms of the Boolean algebra of central elements of $\mathcal{L}$ reflects the fact that one can simultaneously predicate about all possibilities because Boolean homomorphisms of the form $v : Z(\mathcal{L}) \to \mathbf{2}$ can be always established. If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \Diamond P$. Classical consequences that are compatible with a given property, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible property. This is interpreted as, if P is a property of the system, $\Diamond P$ is the smallest central element greater than P, i.e. $\Diamond P = Min\{z \in Z(\mathcal{L}) : P \leq z\}$. This enriched orthomodular structure called *Boolean saturated orthomodular lattices* can be axiomatized by equations conforming a variety denoted by $\mathcal{OML}^\Diamond$ [4, Theorem 4.5]. Orthomodular complete lattices are examples of Boolean saturated orthomodular lattices. We can embed each orthomodular lattice $\mathcal{L}$ in an element $\mathcal{L}^\Diamond \in \mathcal{OML}^\Diamond$ see [4, Theorem 10]. The *modal extension* of $\mathcal{L}$, namely $\mathcal{L}^\Diamond$, represents the fact that each orthomodular system can be modally enriched in such a way as to obtain a new propositional system that includes the original propositions in addition to their possibilities. Let $\mathcal{L}$ be an orthomodular lattice and $\mathcal{L}^\Diamond$ a modal extension of $\mathcal{L}$. We define the possibility space of $\mathcal{L}$ in $\mathcal{L}^\Diamond$ as as the subalgebra of $\mathcal{L}^\Diamond$ generated by the set $\{\Diamond(P) : P \in \mathcal{L}\}$. This algebra is denoted by $\Diamond\mathcal{L}$ and we can prove that it is a Boolean subalgebra of the modal extension. Even though the modal extension $\mathcal{L}^\Diamond$ of $\mathcal{L}$ represents the complete propositional system. The possibility space represents a classical structure in which only the possibilities added to the discourse about properties of the system are organized. Within this frame, the actualization of a possible property acquires a rigorous meaning. Let $\mathcal{L}$ be an orthomodular lattice, $(W_i)_{i\in I}$ the family of Boolean sublattices of $\mathcal{L}$ and $\mathcal{L}^\Diamond$ a modal extension of $\mathcal{L}$. If $f : \Diamond\mathcal{L} \to \mathbf{2}$ is a Boolean homomorphism, an actualization compatible with f is a global valuation $(v_i : W_i \to \mathbf{2})_{i\in I}$ such that $v_i \mid W_i \cap \Diamond\mathcal{L} = f \mid W_i \cap \Diamond\mathcal{L}$ for each $i \in I$. *Compatible actualizations* represent the passage from possibility to actuality, they may be regarded as formal constrains when applying the interpretational rules proposed in the different modal versions. When taking into account compatible actualizations from different contexts, an analogous of the KS theorem — which we have called Modal Kochen Specker (MKS) for obvious reasons— holds for possible properties.

Theorem 4.2 [4, Theorem 6.2] *Let $\mathcal{L}$ be an orthomodular lattice. Then $\mathcal{L}$ admits a global valuation iff for each possibility space there exists a Boolean homomorphism $f : \Diamond\mathcal{L} \to \mathbf{2}$ that admits a compatible actualization.* □

The MKS theorem shows that no enrichment of the orthomodular lattice with modal propositions allows to circumvent the contextual character of the quantum language. Thus, from a formal perspective, one is forced to conclude that quantum possibility is something different from classical possibility.

The larger structure allows to compare the classical and quantum cases. In the classical case, the elements $A \in \wp(\Gamma)$ interpreted as the properties of the system are part of a Boolean algebra (with Γ the classical phase space and $\wp(\Gamma)$ its power set). The elements of the corresponding modal structure are constructed by applying the possibility operator $\Diamond$ to the elements A. These new elements $\Diamond A$, that belong to the modal structure, correspond to possible properties as spoken in the natural language. However, in this case, the seemingly larger structure that includes both actual and modal propositions does not enlarge the expressive power of the language. This is due to the fact that there exists a trivial correspondence between any pair of classical valuations v_c and w_c of the actual and the possible structures to truth-falsity. This relation can be written as follows: let $A_k \in \wp(\Gamma)$, k a fix index, then

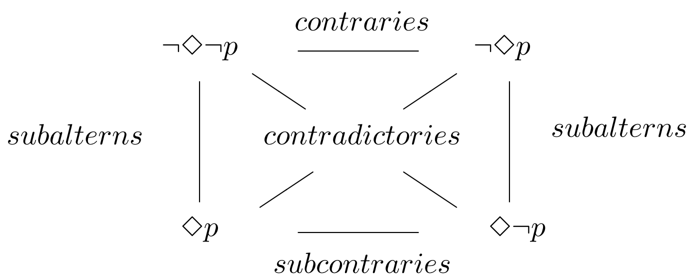
$$w_c(\Diamond A_k) = 1 \Leftrightarrow v_c(A_k) = 1$$

$$w_c(\Diamond A_k) = 0 \Leftrightarrow v_c(A_k) = 0$$

Thus, given the state of a classical system, possible properties at a certain time coincide with (simultaneous) actual ones, they may be identified. And the distinction between the two sets of properties is never made. In fact, when referring to possible properties of a system in a given state, one is always making reference to *future* possible values of the magnitudes, values that are determined because they are the evaluation of functions at points $(p,\ q)$ in Γ at future times. These points are determined by the equation of motion. Thus, not even future possibilities are classically indeterminate and they coincide with *future actual properties.* On the contrary, in the quantum case, the projectors $\mathbf{P}_a = |a\rangle\langle a|$ on $\mathcal{H}$, which are interpreted as the properties of a system, belong to an orthomodular structure. As we have mentioned above, the orthomodular lattice is enlarged with its modal content by adding the elements $\Diamond_Q|a\rangle\langle a|$. Due to the fact that there is no trivial relation between the valuations of subsets of the possible and actual elements to truth-falsity, this new structure genuinely enlarges the expressive power of the language. Formally, if $w_q(\Diamond_Q \mathbf{P}_k) = 1$, with $\mathbf{P}_k \in W_i$, then there exists a valuation v_q such that $v_q(\mathbf{P}_k) = 1$ and another v'_q such that $v'_q(\mathbf{P}_k) = 0$. Thus, contrary to the classical case, even at the same instant of time, we may consider two different kind of properties, two different realms, possible and actual, that do not coincide.

5 $\mathcal{OML}^{\Diamond}$-Square of Opposition

As we have previously discussed, the restriction of the notion of potentiality to that of logical possibility has been of great importance for the development of modern science. The need to interpret QM suggests the reconsideration of this notion in the light of its non-classical structure. In order to do so, we have studied the Aristotelian square of opposition in $\mathcal{OML}^{\Diamond}$. Such a version of the Square of Opposition is also called *Modal Square of Opposition* (MSO) and expresses the essential properties of the monadic first order quantifiers $\forall$, $\exists$. In an algebraic approach, these properties can be represented within the frame of monadic Boolean algebras [8]. More precisely, quantifiers are considered as modal operators acting on a Boolean algebra while the MSO is represented by relations between certain terms of the language in which the algebraic structure is formulated.



The interpretations given to $\Diamond$ from different modal logics determine the corresponding versions of the MSO and by changing the underlying Boolean structure we obtain several generalizations of the monadic first order logic.

In what follows we shall interpret this MSO in $\mathcal{OML}^{\Diamond}$. This version of the MSO will be referred as $\mathcal{OML}^{\Diamond}$-Square of Opposition.

Let $\mathcal{L}$ be an orthomodular lattice and $p \in \mathcal{L}$ such that $p \notin Z(\mathcal{L})$, i.e. p can be seen as a non classical proposition in a quantum system represented by $\mathcal{L}$. Let $\mathcal{L}^{\Diamond}$ be a modal extension of $\mathcal{L}$, W be a Boolean subalgebra of $\mathcal{L}$, i.e. a context, such that $p \in W$ and consider a classically expanded context $W^{\Diamond}$ defined as the sub-algebra of $\mathcal{L}^{\Diamond}$ generated by $W \cup Z(\mathcal{L}^{\Diamond})$. To analyze the Square, first of all we recall that $\neg p$ is the orthocomplement of p. Thus, $\neg$ does not act as a classical negation. But, when applied to possible properties $(\neg\Diamond p)$, $\neg$ acts as a classical negation since $\Diamond p$ is a central element.

- $\neg\Diamond\neg p$ <u>*contraries*</u> $\neg\Diamond p$

Contrary proposition is the negation of the minimum classical consequence of $\neg p$ (the orthogonal complement of p) with respect to the negation of the minimum classical consequence of p. In the usual explanation, two propositions are contrary iff they cannot both be true but can both be false. In our framework we can obtain a similar concept of contrary propositions. Note that $(\neg\Diamond\neg p) \wedge (\neg\Diamond p) \leq p \wedge \neg p = 0$. Hence there is not a Boolean valuation $v : W^{\Diamond} \to \mathbf{2}$ such that $v(\neg\Diamond\neg p) = v(\neg\Diamond p) = 1$, i.e. $\neg\Diamond\neg p$ and $\neg\Diamond p$ "cannot both be true" in each possible classically expanded context $W^{\Diamond}$. Since $p \notin Z(\mathcal{L})$, it is not very hard to see that $\Diamond p \wedge \Diamond\neg p \neq 0$. Then there exists a Boolean valuation $v : W^{\Diamond} \to \mathbf{2}$ such that $v(\Diamond p \wedge \Diamond\neg p) = 1$. Thus $0 = \neg v(\Diamond p \wedge \Diamond\neg p) = v(\neg\Diamond p) \vee v(\neg\Diamond\neg p)$. Hence $\neg\Diamond p$ and $\neg\Diamond\neg p$ can both be false.

- $\Diamond p$ <u>*subcontraries*</u> $\Diamond\neg p$

The sub-contrary proposition is the smallest classical consequence of p with respect to the smallest classical consequence of $\neg p$. Note that sub-contrary propositions do not depend on the context. In the usual explanation, two propositions are sub-contrary iff they cannot both be false but can both be true. Suppose that there exists a Boolean homomorphism $v : W^{\Diamond} \to \mathbf{2}$ such that $v(\Diamond p) = v(\Diamond\neg p) = 0$. Since $p \leq \Diamond p$ and $\neg p \leq \Diamond\neg p$ then $v(p) = v(\neg p) = 0$ which is a contradiction. Then they cannot both be false. Since $p \notin Z(\mathcal{L})$, it is not very hard to see that $\Diamond p \wedge \Diamond\neg p \neq 0$. Hence there exists a Boolean homomorphism $v : W^{\Diamond} \to \mathbf{2}$ such that $1 = v(\Diamond p \wedge \Diamond\neg p) = v(\Diamond p) \wedge v(\Diamond\neg p)$. Then $\Diamond p$ and $\Diamond\neg p$ can both be true.

- $\neg\Diamond\neg p$ <u>*subalterns*</u> $\Diamond p$ and $\neg\Diamond p$ <u>*subalterns*</u> $\Diamond\neg p$

We study the subalterns propositions $\neg\Diamond\neg p$ and $\Diamond p$ since the other case is analog. For our case, a subaltern proposition is the negation, in the classical sense, of the minimum classical consequence of $\neg p$ (the orthogonal complement of p) with respect to the minimum classical consequence of p. In the usual explanation, a proposition is subaltern to another one, called *superaltern*, iff it must be true when its superaltern is true and the superaltern must be false when its subaltern is false. In our case $\neg\Diamond\neg p$ is superaltern of $\Diamond p$ and $\neg\Diamond p$ is superaltern of $\Diamond\neg p$. Since $\neg\Diamond\neg p \leq p \leq \Diamond p$, for each valuation $v : W^{\Diamond} \rightarrow \mathbf{2}$, if $v(\neg\Diamond\neg p) = 1$ then $v(\Diamond p) = 1$ and if if $v(\Diamond p) = 0$ then $v(\neg\Diamond\neg p) = 0$ as is required in the subalterns propositions.

- $\neg\Diamond\neg p$ <u>*contradictories*</u> $\Diamond\neg p$ and $\Diamond p$ <u>*contradictories*</u> $\neg\Diamond p$

The notion of contradictory propositions can be reduced to the relation between $\Diamond p$ and $\neg\Diamond p$. Contradictory propositions are the minimum classical consequence of p with respect to the negation of its minimum classical consequence. In the usual explanation, two propositions are contradictory iff they cannot both be true and they cannot both be false. Since $\Diamond p$ is a central element, this property is trivially maintained for $\Diamond p$ and $\neg\Diamond p$.

We wish to remark that in terms of valuations the $\mathcal{OML}^{\Diamond}$-Square of Opposition behaves in analogous manner to the traditional Square of Opposition, the essential difference being that the concept of "opposed proposition" represented in the $\mathcal{OML}^{\Diamond}$-Square of Opposition is rooted in the concept of minimum classical consequence of a property of a physical system. This fact manifests itself in the following: if p were a classic proposition, i.e. $p \in Z(\mathcal{L})$, then $p = \Diamond p$. In other words, the minimum classical consequence of p is itself. In this way the concept of minimum classical consequence is trivialized, and consequently, also the Square. More precisely, on the one hand the concepts of contradictories, contraries and subcontraries propositions collapse with the classical contradiction $\{p, \neg p\}$ and the subalternation collapses to the trivial equivalence $p \leftrightarrow p$.

The previous analysis exposes once again the fact that classical possibility and quantum possibility formally behave in different manners. This argument adds to the discussion provided in [17] calling the attention to the misinterpretation of the notion of possibility in QM.

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